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A class of integrable lattices and KP hierarchy

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Abstract

We introduce a class of integrable l-field first-order lattices together with corresponding Lax equations. These lattices may be represented as a consistency condition for auxiliary linear systems defined on sequences of formal dressing operators. This construction provides a simple way to build lattice Miura transformations between one-field lattice and l-field ($l \geqslant 2$) ones. We show that the lattices pertained to by the above class are in some sense compatible with KP flows and define the chains of constrained Kadomtsev–Petviashvili Lax operators.

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1. Introduction

In recent times, the Kadomtsev–Petviashvili (KP) hierarchy and its possible reductions has drawn much attention because of the vast variety of applications to different branches of physics. Recall that in the Sato framework the KP hierarchy is given by the Lax equation

$$\partial_p \mathcal{Q} = [(\mathcal{Q}^p)_+, \mathcal{Q}] \tag{1}$$

on the first-order pseudo-differential operator (ΨDO) $\mathcal{Q} = \partial + \sum_{k=1}^{\infty} U_k(\underline{t})\partial^{-k}$. An equivalent form of the Lax equation (1) is the Sato–Wilson equation

$$\partial_p \hat{w} = -(\hat{w}\partial^p \hat{w}^{-1})_- \hat{w} = (\hat{w}\partial^p \hat{w}^{-1})_+ \hat{w} - \hat{w}\partial^p \tag{2}$$

with the formal dressing operator $\hat{w}$ being defined through $\mathcal{Q} = \hat{w}\partial\hat{w}^{-1}$. Other objects associated with $\mathcal{Q}$ are the formal Backer–Akhiezer wavefunction ψ and its conjugate ψ^* defined by $\psi(\underline{t}, z) = \hat{w}\exp(\xi(\underline{t}, z))$ and $\psi^*(\underline{t}, z) = \hat{w}^{*-1}\exp(-\xi(\underline{t}, z))$, respectively, with $\xi(\underline{t}, z) = \sum_{p=1}^{\infty} t_p z^p$.

A very important observation is that the KP wavefunctions satisfy the bilinear identity [1]

$$\mathrm{res}_z \psi(\underline{t}, z)\psi^*(\underline{t}', z) = 0 \tag{3}$$

providing a description of the KP hierarchy in terms of Hirota's bilinear equations on the τ-function. After introducing $\tau(\underline{t})$ through

$$\psi(\underline{t}, z) = \frac{\tau(\underline{t} - [z^{-1}])}{\tau(\underline{t})} \exp(\xi(\underline{t}, z)) \qquad \psi^*(\underline{t}, z) = \frac{\tau(\underline{t} + [z^{-1}])}{\tau(\underline{t})} \exp(-\xi(\underline{t}, z))$$

identity (3) becomes

$$\mathrm{res}_z \tau(\underline{t} - [z^{-1}])\tau(\underline{t}' + [z^{-1}]) \exp(\xi(\underline{t} - \underline{t}', z)) = 0$$

with $[z^{-1}] = (1/z, 1/(2z^2), 1/(3z^3), \ldots)$.

In this paper we suggest a construction which provides a relationship between some class of multi-field lattices and chains of constrained KP (cKP) Lax operators. Recently cKP hierarchies have received a lot of interest because of the strong relationship with multi-matrix models in non-perturbative string theory [4, 5]. It is known that cKP hierarchies contain a number of interesting (from a physical point of view) integrable nonlinear evolution equations whose applications range from hydrodynamics to modern theories in high-energy elementary particle physics. An important property of cKP hierarchies is the existence of a discrete symmetry structure defined by successive Darboux–Bäcklund transformations of suitable Lax operators [5]. The integrable lattices play a key role in this and serve as a source of discrete symmetries for evolution-type systems [6–8] by 'gluing' copies of the equations of cKP. In particular, the relationship between Toda and Volterra lattices and some class of cKP hierarchies is well known.

In this paper we show a class of integrable l-field lattices. These are defined as the consistency condition of some auxiliary linear systems. In fact these auxiliary systems define compatible pairs of shifts s_1 and s_2 on sequences of dressing operators $\{\hat{w}_i, \in \boldsymbol{Z}\}$. This construction naturally lets one build infinite sets of constrained Lax operators which are 'glued' together by compatible pairs of similarity gauge transformations. The important property of these lattices is that in a sense they are consistent with the KP flows given by the Lax equation (1) or the Sato–Wilson equation (2).

In many ways this paper was influenced by the work [2] where the system

$$(\partial + u_i)\hat{w}_i = \hat{w}_{i+1}\partial$$
$$\partial_p \hat{w}_i = -(\mathcal{Q}_i^p)_- \hat{w}_i \qquad \text{where} \quad \mathcal{Q}_i = \hat{w}_i \partial^p \hat{w}_i^{-1}$$
$$\partial_p u_i = (\mathcal{Q}_{i+1}^p)(\partial + u_i) - (\partial + u_i)(\mathcal{Q}_i^p) \qquad i \in \boldsymbol{Z}$$

was introduced and referred to as the modified KP hierarchy. It was shown that the modified KP hierarchy is in fact equivalent to the discrete KP (one-dimensional Toda lattice) hierarchy. The Lax operators $\mathcal{Q}_i$ in this case are unconstrained and only connected with each other by similarity gauge transformations. Our construction builds sequences of cKP copies, which is achieved by imposing two compatible auxiliary linear constraints on the sequences $\{\hat{w}_i,\ i \in \boldsymbol{Z}\}$. The compatibility of the latter is guaranteed by the integrable lattice equations. Another important inspiration of this work is Krichever's rational reductions of the KP hierarchy [3] (or cKP as this is commonly called).

A scheme of the paper is as follows. In section 2 we present integrable chains. In section 3 we show how one can construct lattice Miura transformations and provide the reader with some examples. We also give the Miura transformations in explicit form for two-field lattices. In section 4 we show the compatibility of the lattices with KP flows.

2. Integrable lattices

The main objective of this section is to define a class of integrable lattices. The term 'integrable' here means only that the given lattice admits a Lax representation. We will consider first-order differential-difference systems on finite numbers of fields $a_k(i,x)$ as being functions of the discrete variable $i \in \boldsymbol{Z}$ and the continuous variable $x \in \boldsymbol{R}^1$.

Before proceeding further let us define the notation used. The unknown functions (fields) depend on the spatial variable $x \in \boldsymbol{R}^1$ and some evolution parameters t_p. We use the short-hand notation $\underline{t}$ to denote an infinite collection of independent variables $(x, t_2, t_3, \ldots)$. The symbols ∂ and ∂_p stand for derivation with respect to x and t_p, respectively. Let A be a ΨDO $A = \sum_{i=-\infty}^{N} a_i(x)\partial^i$, of order N. The subscripts '+' and '−' mean the differential and integral parts of A, respectively. We write $\partial_p A$ to denote the derivative of ΨDO A with respect to t_p (not a product).

For an arbitrary pair of integers $n \in \boldsymbol{N}$ and $m \leqslant n-1$ we define an infinite collection of first-order differential operators

$$H_i = \partial - \sum_{k=1}^{n} a_0(i+k-1,x) \qquad i \in \boldsymbol{Z} \tag{4}$$

and ΨDO values

$$G_i = \partial + \sum_{k=1}^{|m|} a_0(i-k,x) + \sum_{k=1}^{l-1} a_k(i+m,x) H_{i-kn}^{-1} \ldots H_{i-2n}^{-1} H_{i-n}^{-1}$$

for $m \leqslant -1$ and

$$G_i = \partial - \sum_{k=1}^{m} a_0(i+k-1,x) + \sum_{k=1}^{l-1} a_k(i+m,x) H_{i-kn}^{-1} \ldots H_{i-2n}^{-1} H_{i-n}^{-1}$$

for $m = 0, \ldots, n-1$. Notice that the definitions of H_i and G_i involve a finite number of fields $\{a_0(i,x), a_1(i,x), \ldots, a_{l-1}(i,x)\}$.

Let us define the following auxiliary equations on an infinite collection of dressing operators $\{\hat{w}_i,\ i \in \boldsymbol{Z}\}$:

$$G_i \hat{w}_i = \hat{w}_{i+m}\partial \qquad H_i \hat{w}_i = \hat{w}_{i+n}\partial. \tag{5}$$

Obviously the latter can be rewritten in terms of BA functions as

$$G_i \psi_i = z\psi_{i+m} \qquad H_i \psi_i = z\psi_{i+n}. \tag{6}$$

The linear system (6) (or (5)) is overdetermined but one can show that the compatibility conditions of (6) are a well determined system of equations for the fields $a_k(i,x)$.

Formally, the consistency condition of (6) is given by

$$G_{i+n} H_i = H_{i+m} G_i \tag{7}$$

or equivalently as $H_{i+m}^{-1} G_{i+n} = G_i H_i^{-1}$. Relation (7) is not convenient for use in further calculations. A technical observation which is helpful in this situation is that (6) can be rewritten in terms of a $(\boldsymbol{L}, \boldsymbol{A})$ pair

$$\boldsymbol{L}(\psi_i) = z\psi_i \qquad \psi_i' = \boldsymbol{A}(\psi_i)$$

with $\boldsymbol{L}$ and $\boldsymbol{A}$ being the difference operators acting on the space of sequences of BA functions $\{\psi_i,\ \ i \in \boldsymbol{Z}\}$ as

$$\begin{aligned} \boldsymbol{L}(\psi_i) &= z\psi_{i+n-m} + \left(\sum_{s=1}^{n-m} a_0(i+s-1)\right)\psi_{i-m} + \sum_{j=1}^{l-1} \frac{1}{z^j} a_j(i)\psi_{i-m-jn} \\ \boldsymbol{A}(\psi_i) &= z\psi_{i+n} + \left(\sum_{s=1}^{n} a_0(i+s-1)\right)\psi_i. \end{aligned} \tag{8}$$

Then the consistency condition of (6) is expressed in a form of the Lax equation

$$\boldsymbol{L}' = [\boldsymbol{A}, \boldsymbol{L}] = \boldsymbol{A}\boldsymbol{L} - \boldsymbol{L}\boldsymbol{A}. \tag{9}$$

One can show that the latter holds if the functions $a_k(i, x)$ satisfy the l-field lattice

$$\begin{aligned}
\sum_{s=1}^{n-m} a_0'(i+s-1) &= \sum_{s=1}^{n-m} a_0(i+s-1)\left(\sum_{s=1}^{n} a_0(i+s-1) - \sum_{s=1}^{n} a_0(i+s-m-1)\right) \\
&\quad + a_1(i+n) - a_1(i) \\
a_k'(i) &= a_k(i)\left(\sum_{s=1}^{n} a_0(i+s-1) - \sum_{s=1}^{n} a_0(i+s-m-kn-1)\right) \\
&\quad + a_{k+1}(i+n) - a_{k+1}(i) \qquad k = 1, \ldots, l-1.
\end{aligned} \tag{10}$$

Here it is understood that $a_l(i, x) = 0$.

Example 2.1. Consider the case $n = 1, m = -1, l = 2$. System (10) becomes

$$\begin{aligned}
&a_0'(i) + a_0'(i+1) = (a_0(i) + a_0(i+1))(a_0(i) - a_0(i+1)) + a_1(i+1) - a_1(i) \\
&a_1'(i) = 0.
\end{aligned}$$

So, actually we have in this case a one-field lattice[1]

$$r_i' + r_{i+1}' = r_i^2 - r_{i+1}^2 + \nu_i \tag{11}$$

with $\nu_i = a_1(i+1) - a_1(i)$ being some constants. As is already known the lattice (11) describes the elementary Darboux transformation for the Schrödinger operator $L = \partial^2 - q(x)$. An interesting property of the lattice (11) is that it reduces to the Painlevé transcendents P_4 and P_5 due to imposing the periodicity conditions

$$r_{i+N} = r_i \qquad \nu_{i+N} = \nu_i$$

for $N = 3$ and 4, respectively [9].

Example 2.2. In the case $m = 0,\ n = 1,\ l \geqslant 2$ we obtain the well known generalized Toda systems

$$\begin{aligned}
&a_0'(i) = a_1(i+1) - a_1(i) \\
&a_k'(i) = a_k(i)\,(a_0(i) - a_0(i-k)) + a_{k+1}(i+1) - a_{k+1}(i) \qquad k = 1, \ldots, l-1.
\end{aligned} \tag{12}$$

In particular if $l = 2$ we obtain the ordinary Toda lattice in polynomial form

$$\begin{aligned}
&a_0'(i) = a_1(i+1) - a_1(i) \\
&a_1'(i) = a_1(i)\,(a_0(i) - a_0(i-1)).
\end{aligned} \tag{13}$$

Defining u_i by the relations $a_0(i) = -u_i'$ and $a_1(i) = \exp(u_{i-1} - u_i)$ we arrive at the more familiar exponential form of the Toda lattice $u_i'' = \mathrm{e}^{u_{i-1}-u_i} - \mathrm{e}^{u_i - u_{i+1}}$.

Example 2.3. Let $m = n-1,\ l = 1,\ n \geqslant 2$. This choice corresponds to the Bogoyavlenskii lattices [10]

$$r_i' = r_i\left(\sum_{k=1}^{n-1} r_{i+k} - \sum_{k=1}^{n-1} r_{i-k}\right). \tag{14}$$

[1] For one-field lattices we use the notation $a_0(i) = r_i$.

3. Lattice Miura transformations

The representation of the chains (10) as the consistency condition of auxiliary linear equations (6) allows us to construct simply and algorithmically Miura mapping which connects the solutions of one-field lattices with the solutions of the corresponding l-field ($l \geqslant 2$) ones.

Firstly notice that $F_i = G_{i+(l-1)n} H_{i+(l-2)n} \ldots H_{i+n} H_i$ is a l-order differential operator. As a consequence of (6) we obtain

$$F_i \psi_i = z^l \psi_{i+(l-1)n+m} \qquad H_i \psi_i = z\psi_{i+n}. \tag{15}$$

Here it is important to notice that (15) is in fact equivalent to (6). Indeed one can express G_i as $G_i = F_{i-(l-1)n} H^{-1}_{i-(l-1)n} \ldots H^{-1}_{i-2n} H^{-1}_{i-n}$ and obtain (6) as a consequence of (15).

Define two integers

$$\overline{n} = ln \qquad \overline{m} = (l-1)n + m. \tag{16}$$

It is evident that $\overline{n} \geqslant 2$ and $\overline{m} < \overline{n}$. Let us identify $\psi_i = \overline{\psi}_{li}$, where $\overline{\psi}_i$ are BA functions being determined by the auxiliary linear system

$$\overline{G}_i \overline{\psi}_i = z\overline{\psi}_{i+\overline{m}} \qquad \overline{H}_i \overline{\psi}_i = z\overline{\psi}_{i+\overline{n}} \tag{17}$$

with $\overline{H}_i = \partial - \sum_{k=1}^{\overline{n}} r_{i+k-1}$ and

$$\overline{G}_i = \begin{cases} \partial + \sum\limits_{k=1}^{|\overline{m}|} r_{i-k} & \text{for} \quad \overline{m} \leqslant -1 \\ \partial - \sum\limits_{k=1}^{\overline{m}} r_{i+k-1} & \text{for} \quad \overline{m} \geqslant 1. \end{cases}$$

The consistency condition of the auxiliary equations (17) is equivalent to the one-field lattice

$$\sum_{s=1}^{\overline{n}-\overline{m}} r'_{i+s-1} = \sum_{s=1}^{\overline{n}-\overline{m}} r_{i+s-1} \left(\sum_{s=1}^{\overline{n}} r_{i+s-1} - \sum_{s=1}^{\overline{n}} r_{i+s-\overline{m}-1} \right). \tag{18}$$

As a consequence of the auxiliary equations (17) we have

$$\overline{F}_i \overline{\psi}_i = z^l \overline{\psi}_{i+l\overline{m}} \qquad \overline{H}_i \overline{\psi}_i = z\overline{\psi}_{i+\overline{n}} \tag{19}$$

with $\overline{F}_i = \overline{G}_{i+(l-1)\overline{m}} \ldots \overline{G}_{i+\overline{m}} \overline{G}_i$. Comparing (19) with (15) we arrive at the following identification:

$$F_i = \overline{F}_{li} \qquad H_i = \overline{H}_{li}. \tag{20}$$

Now let us look at some examples of Miura transformations calculated using (20).

Example 3.1. Take for example $\overline{n} = 2$ and $\overline{m} = 1$. Solving (16) gives $n = 1$, $m = 0$ and $l = 2$. In this case we derive the well known relations

$$a_0(i) = r_{2i} + r_{2i+1} \qquad a_1(i) = r_{2i-1} r_{2i} \tag{21}$$

defining a mapping of solutions of the Volterra lattice

$$r'_i = r_i (r_{i+1} - r_{i-1}) \tag{22}$$

into solutions of the Toda lattice (13) [11].

Example 3.2. For the system

$$\begin{aligned} a'_0(i) &= a_1(i+1) - a_1(i) \\ a'_1(i) &= a_1(i)\,(a_0(i) - a_0(i-1)) + a_2(i+1) - a_2(i) \\ a'_2(i) &= a_2(i)\,(a_0(i) - a_0(i-2)) \end{aligned} \tag{23}$$

we obtain the following Miura transformation:

$$a_0(i) = r_{3i} + r_{3i+1} + r_{3i+2}$$
$$a_1(i) = r_{3i-2}r_{3i} + r_{3i-1}r_{3i} + r_{3i-1}r_{3i+1}$$
$$a_2(i) = r_{3i-4}r_{3i-2}r_{3i}.$$

The latter relates (23) to Bogoyavlenskii lattice $r_i' = r_i(r_{i+2} + r_{i+1} - r_{i-1} - r_{i-2})$.

From (16) it follows that the same one-field lattice is connected by Miura transformations, generally speaking, with a number of l-field ones. It is obvious that the number of such lattices is defined by how many divisors of $\overline{n}$ are among $l = 2, \ldots, \overline{n}$.

Example 3.3. Consider the Bogoyavlenskii lattice

$$r_i' = r_i(r_{i+3} + r_{i+2} + r_{i+1} - r_{i-1} - r_{i-2} - r_{i-3}) \tag{24}$$

corresponding to the choice $\overline{n} = 4$ and $\overline{m} = 3$ in (18). Equations (16) in this case have two solutions: $n = 2,\ m = 1,\ l = 2$ and $n = 1,\ m = 0,\ l = 4$. For the first solution of (16) we obtain the Miura transformation

$$a_0(i) = r_{2i} + r_{2i+1} \qquad a_1(i) = r_{2i-3}r_{2i}$$

relating (24) to the two-field system

$$a_0'(i) = a_0(i)(a_0(i+1) - a_0(i-1)) + a_1(i+2) - a_1(i)$$
$$a_1'(i) = a_1(i)(a_0(i+1) + a_0(i) - a_0(i-2) - a_0(i-3)).$$

The second solution of (16) corresponds to the generalized Toda lattice (12) in the case $l = 4$, i.e.

$$a_0'(i) = a_1(i+1) - a_1(i)$$
$$a_1'(i) = a_1(i)\,(a_0(i) - a_0(i-1)) + a_2(i+1) - a_2(i)$$
$$a_2'(i) = a_2(i)\,(a_0(i) - a_0(i-2)) + a_3(i+1) - a_3(i)$$
$$a_3'(i) = a_3(i)\,(a_0(i) - a_0(i-3))\,.$$

The Miura transformation in this case is given by

$$a_0(i) = r_{4i} + r_{4i+1} + r_{4i+2} + r_{4i+3}$$
$$a_1(i) = r_{4i-3}r_{4i} + r_{4i-2}r_{4i+1} + r_{4i-1}r_{4i+2} + r_{4i-2}r_{4i} + r_{4i-1}r_{4i} + r_{4i-1}r_{4i+1}$$
$$a_2(i) = r_{4i-6}r_{4i-3}r_{4i} + r_{4i-5}r_{4i-3}r_{4i} + r_{4i-5}r_{4i-2}r_{4i} + r_{4i-5}r_{4i-2}r_{4i+1}$$
$$a_3(i) = r_{4i-9}r_{4i-6}r_{4i-3}r_{4i}.$$

Let us show the results of the calculations of the Miura transformations for two-field systems. The formulae can be written in unique form:

$$a_0(i) = r_{2i} + r_{2i+1} \qquad a_1(i) = \sum_{s=1}^{n-m} r_{2i+s-n-m-1} \cdot \sum_{s=1}^{n-m} r_{2i+s-1}.$$

Notice that the systems corresponding to $m \leqslant -1$ and $n = |m|$ are excluded from consideration since we have $\overline{m} = 0$. In fact we are dealing here with one-field lattices since $a_1'(i) = 0$.

4. The chains of KP Lax operators

Relations (7) play a key role in defining Lax operators $\mathcal{Q}_i$ connected to each other by a compatible pair of similarity transformations. The subscript $i \in \boldsymbol{Z}$ can be interpreted as a discrete evolution parameter. The principal problem which is naturally raised here is to define equations for the fields $a_k(i) = a_k(i, \underline{t})$ which guarantee the compatibility of the mappings with respect to i with t_p-flows given by the Lax equation (1) or equivalently by the Sato–Wilson equation (2).

Proposition. *By (7) Lax operators are connected with each other by two invertible compatible gauge transformations*

$$\mathcal{Q}_{i+m} = G_i \mathcal{Q}_i G_i^{-1} \tag{25}$$

$$\mathcal{Q}_{i+n} = H_i \mathcal{Q}_i H_i^{-1}. \tag{26}$$

Proof. By (7) we have

$$\begin{aligned} \mathcal{Q}_{i+m} &= \hat{w}_{i+m} \partial \hat{w}_{i+m}^{-1} = (G_i \hat{w}_i \partial^{-1}) \partial (\partial \hat{w}_i^{-1} G_i^{-1}) \\ &= G_i \hat{w}_i \partial \hat{w}_i^{-1} G_i^{-1} = G_i \mathcal{Q}_i G_i^{-1}. \end{aligned} \tag{27}$$

Similar calculations are needed to prove (26). The mapping $\mathcal{Q}_i \to \tilde{\mathcal{Q}}_i = \mathcal{Q}_{i+m}$ we denote as s_1, while s_2 stands for the transformation $\mathcal{Q}_i \to \overline{\mathcal{Q}}_i = \mathcal{Q}_{i+n}$. The compatibility of s_1 and s_2 also follows from (7). Indeed, we obtain

$$\begin{aligned} \mathcal{Q}_{i+n+m} &= G_{i+n} \mathcal{Q}_{i+n} G_{i+n}^{-1} = G_{i+n} H_i \mathcal{Q}_i H_i^{-1} G_{i+n}^{-1} \\ &= H_{i+m} G_i \mathcal{Q}_i G_i^{-1} H_{i+m}^{-1} = H_{i+m} \mathcal{Q}_{i+m} H_{i+m}^{-1}. \end{aligned}$$

and so we can write $s_1 \circ s_2 = s_2 \circ s_1$. The inverse maps s_1^{-1} and s_2^{-1} are well defined by the formulae $\mathcal{Q}_{i-m} = G_{i-m}^{-1} \mathcal{Q}_i G_{i-m}$ and $\mathcal{Q}_{i-n} = H_{i-n}^{-1} \mathcal{Q}_i H_{i-n}$. □

Let p and q be co-prime integers such that $pn = qm$. Without loss of generality we can see that $p \geqslant 0$. It is obvious that the relation $s_1^q = s_2^p$ holds. Indeed the left- and right-hand side of this relation correspond to the same mapping, i.e. $\mathcal{Q}_i \to \mathcal{Q}_{i+pn} = \mathcal{Q}_{i+qm}$. Let us summarize these statements in the following theorem.

Theorem. *Let the collection* $\{a_0(i), a_1(i), \ldots, a_{l-1}(i)\}$ *solve equations of the lattice (10). Then by (7) the set of Lax operators* $\{\mathcal{Q}_i = \hat{w}_i \partial \hat{w}_i^{-1},\ i \in \boldsymbol{Z}\}$ *admits the action of the discrete group* $\mathcal{G}$ *with the pair of generators* s_1 *and* s_2 *realized as gauge transformations (25) and (26). In addition the group elements* s_1 *and* s_2 *are restricted by relations* $s_1 \circ s_2 = s_2 \circ s_1$ *and* $s_1^q = s_2^p$ *where* p *and* q *are co-prime integers such that* $pn = qm$, $p \geqslant 0$.

Let us now suppose that each $\mathcal{Q}_i$ solves the evolution equations of the KP hierarchy (1). Differentiating the left- and right-hand sides of auxiliary equations (5) with respect to t_p by (2) yields the evolution equations

$$\partial_p G_i = (\mathcal{Q}_{i+m}^p)_+ G_i - G_i (\mathcal{Q}_i^p)_+ \tag{28}$$

$$\partial_p H_i = (\mathcal{Q}_{i+n}^p)_+ H_i - H_i (\mathcal{Q}_i^p)_+. \tag{29}$$

Our next goal is to show that the pair of equations (28) and (29) is properly defined and consistent. To prove the correctness of the definition of (29) some standard arguments are needed [2]. One can rewrite (26) as $\mathcal{Q}_{i+n} H_i = H_i \mathcal{Q}_i$. From this follows

$$\mathcal{Q}_{i+n}^p H_i = H_i \mathcal{Q}_i^p$$

for arbitrary $p \in \boldsymbol{N}$. By virtue of this relation one can write

$$(\mathcal{Q}_{i+n}^p)_+ H_i - H_i (\mathcal{Q}_i^p)_+ = -H_i (\mathcal{Q}_{i+n}^p)_- + (\mathcal{Q}_i^p)_- H_i. \tag{30}$$

The right-hand side of (30) is zero-order ΨDO, while the left-hand side of (30) is a purely differential operator. So one conclude that the expression $(\mathcal{Q}_{i+n}^p)_+ H_i - H_i(\mathcal{Q}_i^p)_+$ is a differential operator of zero-order or simply a function.

The situation with (28) is more complicated since G_i values, generally speaking, are ΨDO values in a special form. However, in this situation one can use the equivalent auxiliary system (15), with F_i being, as we have mentioned above, purely a differential operator of the l-order. The evolution equation on F_i follows from (28) and (29) and is

$$\partial_p F_i = (\mathcal{Q}_{i+m+(l-1)n}^p)_+ F_i - F_i(\mathcal{Q}_i^p)_+ \tag{31}$$

whilst the relation

$$\mathcal{Q}_{i+m+(l-1)n} = F_i \mathcal{Q}_i F_i^{-1} \tag{32}$$

is valid. Apparently the gauge transformation (32) corresponds to the group element $s_1 \circ s_2^{l-1} \in \mathcal{G}$. By using the same arguments as for the H_i values one can easily prove that the right-hand side of (31) is a $(l-1)$-order differential operator.

It now remains to prove the correctness of the simultaneous definition of (28) and (29). To do this, it is enough to show that differentiating the left- and right-hand sides of (7) with respect to t_p, by virtue of (28) and (29) gives the identity. It is a straightforward calculation. Let us give some examples of the evolution equations (28) and (29) for t_2-flows.

Example 4.1. Consider the case $n = 2, m = 1, l = 1$ corresponding to the Volterra lattice. t_2 flow for operators $G_i = \partial - r_i$ and $H_i = \partial - r_i - r_{i+1}$ is defined by the evolution equation

$$\partial_2 r_i = (r_i' + r_i^2 + 2r_{i-1}r_i)'. \tag{33}$$

Using $r_i' = r_i(r_{i+1} - r_{i-1})$ from (33) one can obtain the higher counterpart of the Volterra lattice

$$\partial_2 r_i = r_i(r_{i+1}r_{i+2} - r_{i-1}r_{i-2} + r_{i+1}^2 - r_{i-1}^2 + r_i r_{i+1} - r_i r_{i-1}).$$

Example 4.2. In the case $n = 3, m = 1, l = 1$ we obtain

$$\partial_2 r_i = (r_{i-1}' + r_i' + r_i^2 + r_{i-2}r_{i-1} + r_{i-2}r_i + r_{i-1}r_i)'.$$

Notice that if we introduce the variables $s_i = r_i + r_{i+1}$ then by virtue of

$$r_i' + r_{i+1}' = (r_i + r_{i+1})(r_{i+2} - r_{i-1}) \tag{34}$$

the equation

$$\partial_2 s_i = s_i(s_{i+2}s_{i+1} - s_{i-1}s_{i-2}) \tag{35}$$

holds. Notice that (35) can be obtained as a consequence of

$$\partial_2 r_i = s_{i-1}s_i s_{i+1} - s_{i-2}s_{i-1}s_i.$$

Using straightforward calculations one can check that the x- and t_2-flows commute.

The situation in the above example can be generalized as follows. The lattice (35) comes into the well known class of integrable lattices [10]

$$\partial_t s_i = s_i\left(\prod_{k=1}^{n-1} s_{i+k} - \prod_{k=1}^{n-1} s_{i-k}\right) \qquad n \geqslant 2. \tag{36}$$

Remark. It is known that any of the systems (36) can be interpreted as well as Bogoyavlenskii lattices (14) as discrete variants of the Korteweg–de Vries equation [10].

Let $s_i = \sum_{i=1}^{n-1} r_{i+k-1}$. Then one-field lattices corresponding to $m = 1, n \geqslant 2$ can be written as $s_i' = s_i(r_{i+n-1} - r_{i-1})$. Notice that (36) is a consequence of

$$\partial_t r_i = \prod_{k=0}^{n-1} s_{i-k+1} - \prod_{k=0}^{n-1} s_{i-k}.$$

Now with straightforward calculations one can check the commutativity of the x- and t-flows.

Let us show that the systems (36) can be interpreted as restrictions of one-dimensional Toda lattice flows on corresponding invariant manifolds. When $m = 1, n \geqslant 2, l = 1$ the eigenvalue problem $\boldsymbol{L}(\psi_i) = z\psi_i$ takes on the form

$$z\psi_{i+n-1} + s_i\psi_{i-1} = z\psi_i. \tag{37}$$

In the following equations it is convenient to define the new wavefunctions by the relation $\varphi_i = z^i\psi_i$. In terms of φ_i values the auxiliary equation (37) has the following form:

$$\varphi_{i+n-1} + s_i z^{n-1}\varphi_{i-1} = z^{n-1}\varphi_i. \tag{38}$$

Step-by-step one can expand the left-hand side of equation (38) to obtain the eigenvalue problem $\boldsymbol{M}(\varphi_i) = z^{n-1}\varphi_i$, where

$$\boldsymbol{M} = E^{n-1} + \sum_{j=1}^{\infty}\left(\prod_{k=1}^{j} s_{i-j+1}\right)E^{n-j-1} \tag{39}$$

with E being an operator of the elementary shift $E(\eta_i) = \eta_{i+1}$.

For each $n \geqslant 2$ define $\boldsymbol{A}_n = \boldsymbol{M}_+$, where the subscript '+' stands for a projection on the positive part of $\boldsymbol{M}$. The difference operator $\boldsymbol{A}_n$ is used to define the auxiliary evolution equation $\partial\varphi_i/\partial t = \boldsymbol{A}_n(\varphi_i)$. It is easy to verify that the latter is consistent with (38) provided that equation (36) is satisfied. From this one can conclude that the Lax equation $\partial_t\boldsymbol{M} = [\boldsymbol{A}_n, \boldsymbol{M}]$ is equivalent to the system (36).

Let us explain how the above is relevant to the one-dimensional Toda lattice hierarchy [12]. For the Lax operator $\boldsymbol{Q} = E + \sum_{k\geqslant 0} q_k(i)E^{-k}$ one defines the restriction $\boldsymbol{Q}^{n-1} = \boldsymbol{M}$, where $\boldsymbol{M}$ is in (39). This implies that some algebraic constraints on the coefficients $q_k(i)$ must be imposed. For example, in the simplest case of the Volterra lattice ($n = 2$) these constraints are given by

$$q_k(i) = s_{i-k}\ldots s_{i-1}s_i = q_0(i-k)\ldots q_0(i-1)q_0(i) \qquad k \geqslant 1.$$

To conclude this section, let us consider one example which illustrates the relationship between a class of the lattices (10) and other known lattices. In the case $n = 1, m = -1, l = 3$ equations (10) take on the form

$$\begin{aligned} a_0'(i) + a_0'(i+1) &= a_0^2(i) - a_0^2(i+1) + a_1(i+1) - a_1(i) \\ a_1'(i) &= a_2(i+1) - a_2(i) \\ a_2'(i) &= a_2(i)(a_0(i) - a_0(i-1)). \end{aligned} \tag{40}$$

Using (20) one calculates the Miura transformation between (40) and (34) to obtain

$$\begin{aligned} a_0(i) &= r_{3i} + r_{3i+1} + r_{3i+2} \\ a_1(i) &= (r_{3i-1} + r_{3i})(r_{3i} + r_{3i+1}) \\ &\quad + (r_{3i} + r_{3i+1})(r_{3i+1} + r_{3i+2}) + (r_{3i+1} + r_{3i+2})(r_{3i+2} + r_{3i+3}) \\ a_2(i) &= (r_{3i-2} + r_{3i-1})(r_{3i-1} + r_{3i})(r_{3i} + r_{3i+1}). \end{aligned} \tag{41}$$

The higher counterpart of (40) is nothing but the Blaszak–Marciniak lattice [13]

$$\begin{aligned} \partial_2 p_i &= u_{i+2} - u_i \\ \partial_2 v_i &= p_i u_{i+1} - u_i p_{i-1} \\ \partial_2 u_i &= u_i(v_i - v_{i-1}) \end{aligned} \tag{42}$$

with $\partial_2 a_0(i) = u_{i+1} - u_i$. Here we denote $p_i = a_0(i) + a_0(i+1)$, $v_i = a_1(i)$ and $u_i = a_2(i)$. From (41) we easily obtain the Miura transformation

$$\begin{aligned} p_i &= s_{3i} + s_{3i+2} + s_{3i+4} \\ v_i &= s_{3i-1}s_{3i} + s_{3i}s_{3i+1} + s_{3i+1}s_{3i+2} \\ u_i &= s_{3i-2}s_{3i-1}s_{3i} \end{aligned}$$

between (42) and (35).

5. Conclusion

Starting from auxiliary linear equations (6), we have defined a class of integrable first-order l-field lattices. The main feature shared by these is a compatibility with KP flows. This can be exploited by searching for solutions to the above lattices and their higher counterparts in terms of the KP τ-functions. Future research will involve continuing our activities in this direction. It will be of interest to investigate topics such as lattice Bäcklund transformations and nonlinear superposition formulae.

The results of the paper might be of potential interest in the investigation of cKP hierarchies. We believe that any integrable chains (up to Miura transformations) presented in the paper underlie some differential integrable hierarchies with s_1 and s_2 being discrete symmetries. More activity in this direction is in [14] (see also [15]) where we have constructed a modified version of Krichever's rational reductions of the KP hierarchy.

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